E-CLP SOR derivative calculations

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Let the initial asset balances be x_0 and y_0. A swap either swaps an amount x_{in} for the amount y_{out}. Or vice versa swaps an amount y_{in} for the amount x_{out}.

Each swap has a fee γ. Let $f := 1 - \gamma$. Given exact x_{in}, the fee means that the actual swap is on $f x_{\text{in}}$ (similarly for exact y_{in}). Given exact amount out, the fee means that the swap requires an input of $\frac{\text{amount in sans fees}}{f}$.

Given exact x_{out}, calculate y_{in} Here

$$y_{\text{in}} = \frac{1}{f}\left(-y_0 + \frac{-(x_0 - x_{\text{out}} - a)\underline{\lambda} sc - \sqrt{r^2(1-\underline{\lambda} s^2) - (x_0 - x_{\text{out}} - a)^2/\lambda^2}}{1 - \underline{\lambda} s^2} + b\right)$$

And

$$\frac{dy_{\text{in}}}{dx_{\text{out}}} = \frac{1}{f(1-\underline{\lambda} s^2)}\left(\underline{\lambda} sc - \frac{x_0 - x_{\text{out}} - a}{\lambda^2 \sqrt{r^2(1-\underline{\lambda} s^2) - (x_0 - x_{\text{out}} - a)^2/\lambda^2}}\right)$$

This is the `spotPriceAfterSwap` for Balancer SOR.

The square root in the denominator can be calculated with good precision using the methods used in the contract calculation. This should help with overall numerical imprecision. This said, there may be other sources of imprecision (e.g., from a and in multiplications and divisions).

And the `derivativeSpotPriceAfterSwap` is the derivative of this as a function of x_{out}, i.e.,

$$\frac{d^2 y_{\text{in}}}{dx_{\text{out}}^2} = \frac{1}{f(1-\underline{\lambda} s^2)}\Bigg(\frac{1}{\lambda^2 \sqrt{r^2(1-\underline{\lambda} s^2) - (x_0 - x_{\text{out}} - a)^2/\lambda^2}} + \frac{(x_0 - x_{\text{out}} - a)^2}{\lambda^4 (r^2(1-\underline{\lambda} s^2) - (x_0 - x_{\text{out}} - a)^2/\lambda^2)^{3/2}}\Bigg)$$

Given exact y_{out}, calculate x_{in} This is parallel to the previous case, we just need to transpose $y_o \to x_0$, $x_0 \to y_0$, $s \to c$, $c \to s$, $a \to b$, $b \to a$. Then

$$\frac{dx_{\text{in}}}{dy_{\text{out}}} = \frac{1}{f(1-\underline{\lambda}c^2)}\left(\underline{\lambda}sc - \frac{y_0 - y_{\text{out}} - b}{\lambda^2\sqrt{r^2(1-\underline{\lambda}c^2) - (y_0 - y_{\text{out}} - b)^2/\lambda^2}}\right)$$

This is the `spotPriceAfterSwap` for Balancer SOR.

And the `derivativeSpotPriceAfterSwap` is

$$\frac{d^2x_{\text{in}}}{dy_{\text{out}}^2} = \frac{1}{f(1-\underline{\lambda}c^2)}\Big(\frac{1}{\lambda^2\sqrt{r^2(1-\underline{\lambda}c^2) - (y_0 - y_{\text{out}} - b)^2/\lambda^2}} + \frac{(y_0 - y_{\text{out}} - b)^2}{\lambda^4(r^2(1-\underline{\lambda}c^2) - (y_0 - y_{\text{out}} - b)^2/\lambda^2)^{3/2}}\Big)$$

Given exact x_{in}, calculate y_{out} Here

$$y_{\text{out}} = y_0 - \left(\frac{-(x_0 + fx_{\text{in}} - a)\underline{\lambda}sc - \sqrt{r^2(1-\underline{\lambda}s^2) - (x_0 + fx_{\text{in}} - a)^2/\lambda^2}}{1-\underline{\lambda}s^2} + b\right)$$

And

$$\frac{dy_{\text{out}}}{dx_{\text{in}}} = \frac{f}{1-\underline{\lambda}s^2}\left(\underline{\lambda}sc - \frac{x_0 + fx_{\text{in}} - a}{\lambda^2\sqrt{r^2(1-\underline{\lambda}s^2) - (x_0 + fx_{\text{in}} - a)^2/\lambda^2}}\right)$$

The `spotPriceAfterSwap` for Balancer SOR is then $\frac{1}{\text{this}}$.

The `derivativeSpotPriceAfterSwap` is the derivative of `spotPriceAfterSwap` as a function of x_{in}, namely

$$(1-\underline{\lambda}s^2)\cdot\frac{\frac{1}{\lambda^2\sqrt{r^2(1-\underline{\lambda}s^2)-(x_0+fx_{\text{in}}-a)^2/\lambda^2}} + \frac{(x_0+fx_{\text{in}}-a)^2}{\lambda^4(r^2(1-\underline{\lambda}s^2)-(x_0+fx_{\text{in}}-a)^2/\lambda^2)^{3/2}}}{\left(\underline{\lambda}sc - \frac{x_0+fx_{\text{in}}-a}{\lambda^2\sqrt{r^2(1-\underline{\lambda}s^2)-(x_0+fx_{\text{in}}-a)^2/\lambda^2}}\right)^2}$$

Given exact y_{in}, calculate x_{out} This is parallel to the previous case, we just need to transpose $y_o \to x_0$, $x_0 \to y_0$, $s \to c$, $c \to s$, $a \to b$, $b \to a$. Then

$$\frac{dx_{\text{out}}}{dy_{\text{in}}} = \frac{f}{1-\underline{\lambda}c^2}\left(\underline{\lambda}sc - \frac{y_0 + fy_{\text{in}} - b}{\lambda^2\sqrt{r^2(1-\underline{\lambda}c^2) - (y_0 + fy_{\text{in}} - b)^2/\lambda^2}}\right)$$

The `spotPriceAfterSwap` for Balancer SOR is then $\frac{1}{\text{this}}$.

The `derivativeSpotPriceAfterSwap` is

$$(1-\underline{\lambda}c^2)\frac{\frac{1}{\lambda^2\sqrt{r^2(1-\underline{\lambda}c^2)-(y_0+fy_{\mathsf{in}}-b)^2/\lambda^2}}+\frac{(y_0+fy_{\mathsf{in}}-b)^2}{\lambda^4(r^2(1-\underline{\lambda}c^2)-(y_0+fy_{\mathsf{in}}-b)^2/\lambda^2)^{3/2}}}{\left(\underline{\lambda}sc-\frac{y_0+fy_{\mathsf{in}}-b}{\lambda^2\sqrt{r^2(1-\underline{\lambda}c^2)-(y_0+fy_{\mathsf{in}}-b)^2/\lambda^2}}\right)^2}$$

Normalized Liquidity as a function of y_{in} *Remark on notation:* In this section, we only consider the variant of the operation where y_{in} is given and x_{out} is computed. Thus, $x_{\mathsf{out}} = x_{\mathsf{out}}(y_{\mathsf{in}})$ should always be considered as a function of y_{in}. We write short x'_{out} and x''_{out} for the first and second derivative of x_{out} as a function of y_{in}.

The `normalizedLiquidity` is defined as

$$\frac{1}{2}\cdot\frac{1}{\lim_{y_{\mathsf{in}}\to 0}\frac{d}{dy_{\mathsf{in}}}\frac{y_{\mathsf{in}}}{x_{\mathsf{out}}}}.$$

Observe that the fraction on the right is the effective price of a trade of non-infinitesimal size. The limit is necessary because $\frac{y_{\mathsf{in}}}{x_{\mathsf{out}}}$ is ill-defined at 0.[1]

`normalizedLiquidity` is equal to

$$\frac{1}{1-\underline{\lambda}c^2}\cdot\frac{R\left(\underline{\lambda}sc\lambda^2R-(y_0-b)\right)^2}{\lambda^2R^2+(y_0-b)^2}$$

where $R := \sqrt{r^2(1-\underline{\lambda}c^2)-(y_0-b)^2/\lambda^2}$.

Proof. We have by the quotient rule

$$\frac{d}{dy_{\mathsf{in}}}\frac{y_{\mathsf{in}}}{x_{\mathsf{out}}} = \frac{x_{\mathsf{out}}-y_{\mathsf{in}}x'_{\mathsf{out}}}{x_{\mathsf{out}}^2}.$$

Note that this is indeterminate for $y_{\mathsf{in}} = 0$. We compute the limit $y_{\mathsf{in}} \to 0$ via

[1] The fraction is ill-defined at $y_{\mathsf{in}} = 0$ because here also $x_{\mathsf{out}} = 0$ and the fraction would be 0/0. Instead of the above limit, we could also consider the continuous extension of $\frac{y_{\mathsf{in}}}{x_{\mathsf{out}}}$ to $y_{\mathsf{in}} = 0$, which is realized at $p_x = \frac{dy_{\mathsf{in}}}{dx_{\mathsf{out}}}(0)$. However, we do not know a simple closed form (without case distinction) for this continuous extension, and so we cannot compute the derivative of that extension directly. This is different than for, e.g., the 2-CLP, where such a closed-form expression is easy to find.

two applications of L'Hospital's rule: in the limit $y_{\mathsf{in}} \to 0$ we have

$$\begin{aligned}\frac{x_{\mathsf{out}} - y_{\mathsf{in}} x'_{\mathsf{out}}}{x^2_{\mathsf{out}}} &= \frac{x'_{\mathsf{out}} - x'_{\mathsf{out}} - y_{\mathsf{in}} x''_{\mathsf{out}}}{2x_{\mathsf{out}} x'_{\mathsf{out}}} \\ &= \frac{-y_{\mathsf{in}} x''_{\mathsf{out}}}{2x_{\mathsf{out}} x'_{\mathsf{out}}} \\ &= \frac{-x''_{\mathsf{out}} - y_{\mathsf{in}} x'''_{\mathsf{out}}}{2(x'^2_{\mathsf{out}} + x_{\mathsf{out}} x''_{\mathsf{out}})} \\ &= \frac{-x''_{\mathsf{out}}}{2x'^2_{\mathsf{out}}}.\end{aligned}$$

The first and the third equality are applications of L'Hospital's rule and the last one follows because we consider the limit $y_{\mathsf{in}} \to 0$ and all involved functions are well-defined and continuous.

Write short

$$R(y_{\mathsf{in}}) := \sqrt{r^2(1 - \underline{\lambda} c^2) - (y_0 + f y_{\mathsf{in}} - b)^2/\lambda^2}$$

and note that $R(0) = R$.

We know from above that

$$\begin{aligned}x'_{\mathsf{out}} = \frac{dx_{\mathsf{out}}}{dy_{\mathsf{in}}} &= \frac{f}{1 - \underline{\lambda} c^2} \left(\underline{\lambda} sc - \frac{y_0 + f y_{\mathsf{in}} - b}{\lambda^2 \sqrt{r^2(1 - \underline{\lambda} c^2) - (y_0 + f y_{\mathsf{in}} - b)^2/\lambda^2}} \right) \\ &= \frac{f}{1 - \underline{\lambda} c^2} \left(\underline{\lambda} sc - \frac{y_0 + f y_{\mathsf{in}} - b}{\lambda^2 R(y_{\mathsf{in}})} \right) \\ &= \frac{f}{1 - \underline{\lambda} c^2} \cdot \frac{\underline{\lambda} sc \lambda^2 R(y_{\mathsf{in}}) - (y_0 + f y_{\mathsf{in}} - b)}{\lambda^2 R(y_{\mathsf{in}})}\end{aligned}$$

It remains to calculate $x''_{\mathsf{out}} = \frac{d^2 x_{\mathsf{out}}}{dy^2_{\mathsf{in}}}$, which we have not yet computed; the calculation is similar to that of $\frac{d^2 x_{\mathsf{in}}}{dy^2_{\mathsf{out}}}$ from above, though. We have

$$x''_{\mathsf{out}} = -\frac{f}{1 - \underline{\lambda} c^2} \cdot \frac{f \lambda^2 R(y_{\mathsf{in}}) - (y_0 + f y_{\mathsf{in}} - b)\lambda^2 \left(\frac{d}{dy_{\mathsf{in}}} R(y_{\mathsf{in}}) \right)}{\lambda^4 R(y_{\mathsf{in}})^2}.$$

It is easy to calculate that

$$\frac{d}{dy_{\mathsf{in}}} R(y_{\mathsf{in}}) = -\frac{f(y_0 + f y_{\mathsf{in}} - b)}{\lambda^2 R(y_{\mathsf{in}})}$$

so that

$$x''_{\text{out}} = -\frac{f^2}{1-\underline{\lambda}c^2}\cdot\left(\frac{1}{\lambda^2 R(y_{\text{in}})}+\frac{(y_0+fy_{\text{in}}-b)^2}{\lambda^4 R(y_{\text{in}})^3}\right)$$
$$= -\frac{f^2}{1-\underline{\lambda}c^2}\cdot\frac{\lambda^2 R(y_{\text{in}})^2+(y_0+fy_{\text{in}}-b)^2}{\lambda^4 R(y_{\text{in}})^3}.$$

We now receive

$$\texttt{normalizedLiquidity} = \frac{1}{2}\cdot\frac{1}{\frac{-x''_{\text{out}}}{2x'^2_{\text{out}}}(0)} = \frac{x'^2_{\text{out}}}{-x''_{\text{out}}}(0)$$
$$= \frac{\left(\frac{f}{1-\underline{\lambda}c^2}\cdot\frac{\underline{\lambda}sc\lambda^2 R-(y_0-b)}{\lambda^2 R}\right)^2}{\frac{f^2}{1-\underline{\lambda}c^2}\cdot\frac{\lambda^2R^2+(y_0-b)^2}{\lambda^4R^3}}$$
$$= \frac{1}{1-\underline{\lambda}c^2}\cdot\frac{R\left(\underline{\lambda}sc\lambda^2R-(y_0-b)\right)^2}{\lambda^2R^2+(y_0-b)^2}\ \square$$

Normalized Liquidity Given x_{in} In the symmetric case where y is the token paid out and x is the token paid in, we need to replace x and y as well as a and b and s and c. Thus `normalizedLiquidity` is equal to

$$\frac{1}{1-\underline{\lambda}s^2}\cdot\frac{R\left(\underline{\lambda}sc\lambda^2R-(x_0-a)\right)^2}{\lambda^2R^2+(x_0-a)^2}$$

where $R := \sqrt{r^2(1-\underline{\lambda}s^2)-(x_0-a)^2/\lambda^2}$. Note that the definition of R is different from the definition of R in the previous section.

Old, Now Unused Content (Maybe Outdated)

Slippage and Normalized Liquidity Given x_{out} Assume that x is the token paid out and y is the token paid in.

Remark on notation: In this section, we only consider the variant of the operation where x_{out} is an independent variable and y_{in} is computed. Thus, $y_{\text{in}} = y_{\text{in}}(x_{\text{out}})$ should always be considered as a function of x_{out}. We write short y'_{in} and y''_{in} for the first and second derivative of y_{in} as a function of x_{out}.

Consider some reserve state (x, y). Let $p := y'_{\text{in}}(0)$ be the marginal price including fees at (x, y), where $y'_{\text{in}}(0)$ is y'_{in} evaluated at $x_{\text{out}} = 0$. The *slippage* of a trade of size $x_{\text{out}} > 0$ is defined as the quotient of the average price of the trade and the marginal price at (x, y), minus 1, i.e.,

$$S := \frac{\frac{y_{\mathsf{in}}}{x_{\mathsf{out}}}}{p} - 1.$$

The `normalizedLiquidity` is defined as

$$1 \Big/ \lim_{x_{\mathsf{out}} \to 0} \frac{dS}{dx_{\mathsf{out}}},$$

where the limit is necessary because S is ill-defined at 0.[2]

`normalizedLiquidity` is equal to[3]

$$2 \frac{\underline{\lambda} sc - \frac{x_0 - a}{\lambda^2 R}}{\frac{1}{\lambda^2 R} + \frac{(x_0 - a)^2}{\lambda^4 R^3}}$$

where $R := \sqrt{r^2(1 - \underline{\lambda} s^2) - (x_0 - a)^2/\lambda^2}$.

Proof. Since $p = y'_{\mathsf{in}}(0)$ is independent of x_{out}, we have

$$\lim_{y_{\mathsf{in}} \to 0} \frac{dS}{dy_{\mathsf{in}}} = \frac{1}{y'_{\mathsf{in}}(0)} \cdot \lim_{x_{\mathsf{out}} \to 0} \frac{d}{dx_{\mathsf{out}}} \frac{y_{\mathsf{in}}}{x_{\mathsf{out}}}.$$

We first compute the latter derivative. We have

$$\frac{d}{dx_{\mathsf{out}}} \frac{y_{\mathsf{in}}}{x_{\mathsf{out}}} = \frac{y'_{\mathsf{in}} x_{\mathsf{out}} - y_{\mathsf{in}}}{x_{\mathsf{out}}^2}.$$

Note that this is indeterminate for $x_{\mathsf{out}} = 0$. We compute the limit $x_{\mathsf{out}} \to 0$ via L'Hospital's rule: we have

$$\begin{aligned} \lim_{x_{\mathsf{out}} \to 0} \frac{y'_{\mathsf{in}} x_{\mathsf{out}} - y_{\mathsf{in}}}{x_{\mathsf{out}}^2} &= \lim_{x_{\mathsf{out}} \to 0} \frac{y''_{\mathsf{in}} x_{\mathsf{out}} + y'_{\mathsf{in}} - y'_{\mathsf{in}}}{2x_{\mathsf{out}}} \\ &= \lim_{x_{\mathsf{out}} \to 0} \frac{1}{2} y''_{\mathsf{in}} \\ &= \frac{1}{2} y''_{\mathsf{in}}(0). \end{aligned}$$

[2] S is ill-defined at $x_{\mathsf{out}} = 0$ because then also $y_{\mathsf{in}} = 0$ and S contains the expression $0/0$. Instead of the above limit, we could also consider the continuous extension of S to $x_{\mathsf{out}} = 0$, which is realized at $S(0) = 0$ (it is easy to see that $S \to 0$ for $x_{\mathsf{out}} \to 0$). However, we do not know a simple closed form (without case distinction) for this continuous extension of S, and so we cannot compute the derivative of that extension directly. This is different than for, e.g., the 2-CLP, where such a closed-form expression is easy to find.

[3] The formula could of course be simplified by canceling out a factor $\lambda^2 R$. But I'm not sure if it will make things simpler overall.

where the last equality is because y''_{in} is well-defined and continuous at 0. We now know that

$$\lim_{x_{\text{out}}\to 0}\frac{dS}{dx_{\text{out}}}=\frac{y''_{\text{in}}(0)}{2y'_{\text{in}}(0)}.$$

We know from above that

$$y'_{\text{in}}=\frac{dy_{\text{in}}}{dx_{\text{out}}}=\frac{1}{f(1-\underline{\lambda}s^2)}\left(\underline{\lambda}sc-\frac{x_0-x_{\text{out}}-a}{\lambda^2\sqrt{r^2(1-\underline{\lambda}s^2)-(x_0-x_{\text{out}}-a)^2/\lambda^2}}\right)$$

$$y''_{\text{in}}=\frac{d^2y_{\text{in}}}{dx_{\text{out}}^2}=\frac{1}{f(1-\underline{\lambda}s^2)}\left(\frac{1}{\lambda^2\sqrt{r^2(1-\underline{\lambda}s^2)-(x_0-x_{\text{out}}-a)^2/\lambda^2}}\right.$$
$$\left.+\frac{(x_0-x_{\text{out}}-a)^2}{\lambda^4(r^2(1-\underline{\lambda}s^2)-(x_0-x_{\text{out}}-a)^2/\lambda^2)^{3/2}}\right)$$

and therefore, by evaluating these at $x_{\text{out}}=0$, we receive

$$y'_{\text{in}}(0)=\frac{1}{f(1-\underline{\lambda}s^2)}\left(\underline{\lambda}sc-\frac{x_0-a}{\lambda^2R}\right)$$
$$y''_{\text{in}}(0)=\frac{1}{f(1-\underline{\lambda}s^2)}\left(\frac{1}{\lambda^2R}+\frac{(x_0-a)^2}{\lambda^4R^3}\right).$$

The statement of the theorem now follows by plugging $y'_{\text{in}}(0)$ and $y''_{\text{in}}(0)$ into the formula above and taking the reciprocal.

□

Slippage and Normalized Liquidity Given y_{out} In the symmetric case where y is the token paid out and x is the token paid in, we need to replace x and y as well as a and b and s and c. Thus `normalizedLiquidity` is equal to

$$2\frac{\underline{\lambda}sc-\frac{y_0-b}{\lambda^2R}}{\frac{1}{\lambda^2R}+\frac{(y_0-b)^2}{\lambda^4R^3}}$$

where $R:=\sqrt{r^2(1-\underline{\lambda}c^2)-(y_0-b)^2/\lambda^2}$. Note that the definition of R is different from the definition of R in the previous section.