Prices in Higher-Dimensional and Transformed Constant-Function Market Makers

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Abstract

We present a general framework for computing prices in constant-function automated market makers (CFMMs) based on the derivatives of the defining functions. This is essentially an application of the implicit function theorem but presents a much more convenient and error-resistant way to compute prices compared to manual calculation. We then apply our basic result to discuss how prices change when a CFMM curve is transformed by another function. This is convenient for introducing virtual reserves to increase capital efficiency and it provides an intuitive geometric approach for distorting curves to change how the AMM price reacts to demand and supply.

1 Computing CFMM prices from partial derivatives

Consider a function $f : \mathbb{R}^n \to \mathbb{R}$. f is the level function used in a CFMM to be analyzed in the following and n is the number of assets. For example, for the constant-product market maker (CPMM), $f(x) = \prod_{i=1}^n x_i$. A particular *AMM curve* is a level surface of form $L_k := \{x \in \mathbb{R}^n \mid f(x) = k\}$ where $k \in \mathbb{R}$. WLOG we single out x_n as the *numéraire,* i.e., all prices will be denoted in units of x_n. We are interested in the *prices* $p_i := -\frac{\mathrm{d}x_n}{\mathrm{d}x_i}$, where the derivatives are with respect to the surface L_k. This means that p_i indicates, in marginal terms, by how much, starting at a given point x, the x_n dimension needs to change if the x_i dimension changes by one infinitesimal unit, the other dimensions (except for x_i and x_n) do not change and we want to stay on the surface L_k. Note that, while p_i generally depends on k, $k = f(x)$ is fully determined by the point x. Furthermore, there are in principle choices for f where derivative does not exist (such as $f(x) = 1$), but all functions that are actually used for AMMs *do* have such a derivative if the original point x is valid.

To compute these prices, we consider the partial derivatives $\frac{\partial f}{\partial x_i}$, where the derivatives are now *not* taken with respect to any surface but in the regular sense of varying x_i while keeping all other coordinates constant. These partial derivatives are easy to compute for almost all CFMMs used in practice.[1] The following theorem provides a convenient way of computing the prices p_i.

Theorem 1. *The prices of a CFMM with level function f at a point x are*

$$p_i = \frac{\frac{\partial f}{\partial x_i}}{\frac{\partial f}{\partial x_n}}.$$

Proof. This immediately follows from the implicit function theorem applied to the function $g(x_i, x_n) := f(x_1, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_{n-1}, x_n)$. □

1.1 Calculation Rules

We now discuss some simple calculation rules for prices. Define, as a generalization of p_i, the value $p_{j \text{ per } i} := -\frac{\mathrm{d}x_j}{\mathrm{d}x_i}$. This is the price of asset i measured in units of asset j. Note that $p_i = p_{n \text{ per } i}$. Then the following hold:

1. $p_{j \text{ per } i} = p_i / p_j$
2. $p_{j \text{ per } i} = 1/p_{i \text{ per } j}$.

This follows immediately from the obvious generalization of Theorem 1 to arbitrary numéraires and (then) simple algebraic transformations. In general, we have for all i, j, k that

$$\frac{\mathrm{d}x_j}{\mathrm{d}x_i} = -\frac{\mathrm{d}x_k}{\mathrm{d}x_i} \Big/ \frac{\mathrm{d}x_k}{\mathrm{d}x_j}.$$

1.2 Examples

2-dimensional CPMM. Consider the constant-product market maker (CPMM) in two dimensions. Let $n = 2$ and, to make notation more standard, rename x to t and write $x := t_1$ and $y := t_2$. The invariant of the 2-dimensional CPMM is $f(x, y) = xy$. We obviously have $\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (y, x)$ and thus $p_x = y/x$ as expected.

Higher-dimensional CPMM. Consider the CPMM in any number of dimensions, where $f(x) = \prod_i x_i$. We have $\frac{\partial f}{\partial x_i} = \prod_{j \neq i} x_j$ and thus $p_i = \prod_{j \neq i} x_j / \prod_{j \neq n} x_j = x_n / x_i$ since all other x_j cancel out. Note how the marginal prices are the same as in a 2-dimensional CPMM. This is intuitive because, holding all x_j for $j \neq i, n$ fixed, the invariant is $x_i x_n = k / \prod_{j \neq i,n} x_j$, corresponding to a 2-dimensional CPMM with invariant $k / \prod_{j \neq i,n} x_j$.

[1] An exception here may be StableSwap, which cannot be described well using a CFMM form. Angeris and Chitra (2020) present a CFMM form, but the shape of that form of the AMM is only preserved when its parameters are adjusted based on the level k. This might make it harder to apply our theory.

Example: CPSMM. Consider the constant power-sum market maker (CPSMM) in two dimensions $f(x,y) = x^\delta + y^\delta$, where $\delta := 1-\gamma$ and $\gamma \in [0,1)$ so that $\delta \in (0,1]$. We have $\nabla f = (\delta x^{\delta-1}, \delta y^{\delta-1})$ and thus $p_x = x^{\delta-1}/y^{\delta-1} = x^{-\gamma}/y^{-\gamma} = y^\gamma/x^\gamma$. We can see from the prices that the AMM is just constant-price for $\gamma = 0$ converges to the CPMM for $\gamma \to 1$.[2]

2 Transformations of CFMMs

We now study how prices change when an AMM curve is transformed by another function. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a differentiable function and consider $g := f \circ F$.[3] Again, we are interested in the prices p_i, but this time under g, where we would like to re-use our knowledge about f. Denote by p^f the price vector under f and by p^g the price vector under g. Note that $p^f_n = p^g_n = 1$. The following theorem provides a way of translating prices in f to prices in g.

Theorem 2. *The prices in a CFMM according to $g = f \circ F$ are*

$$p_i^g(x) = \frac{\nabla f(F(x)) \cdot DF(x)e_i}{\nabla f(F(x)) \cdot DF(x)e_n} = \frac{p^f(F(x)) \cdot DF(x)e_i}{p^f(F(x)) \cdot DF(x)e_n}.$$

Here, $DF(x) = \left(\frac{\partial F_i(x)}{x_j}\right)_{i,j}$ is the total derivative (i.e., the Jacobian matrix) of F at x.

In the statement of the theorem, the products in the numerators and denominators, respectively, are scalar products of vectors while the fractions themselves denote ordinary divisions. Observe further that $DF(x)e_i = \left(\frac{\partial F_i(x)}{x_j}\right)_j$ is simply the i-th column of the Jacobian.

Proof of Theorem 2. The first equality follows immediately from the chain rule and Theorem 1. More in detail, we have

$$\frac{\partial g}{\partial x_i} = \nabla f(F(x)) \cdot DF(x)e_i$$

by the multi-dimensional chain rule. Replacing this into Theorem 1 yields the statement. The second equality follows by replacing all terms p_i^f based on Theorem 1: we receive $p^f = 1/\frac{\partial f}{\partial x_n} \cdot \nabla f$ and the scalar factor $\frac{\partial f}{\partial x_n}$ cancels out. □

Remark 1. In the following special cases, the statement of the theorem simplifies further:

[2] This was first noted in the YieldSpace whitepaper, see: `https://yield.is/YieldSpace.pdf`.

[3] Note that the level sets L_k change according to F^{-1} (which is a function if F is invertible or a correspondence if it is not). This is because $g(x) = k$ iff $F(x) \in f^{-1}(k)$ iff $x \in F^{-1}(f^{-1}(k))$. In practice, F often is invertible and this means that the level sets under f are transformed by the function F^{-1} to receive the level sets under g.

- If $F =: A$ is a linear transformation, then $DF(x) = A$, so the above formulas become
$$p_i^g(x) = \frac{p^f(F(x)) \cdot Ae_i}{p^f(F(x)) \cdot Ae_n}.$$
Note that the prices only depend on $F(x)$, but not on x directly anymore. This can considerably simplify the analysis.
- If $F(x) = x+v$ where v is constant (i.e., F encodes an offset), then $DF(x) = I$, the identity matrix. It is easy to see that Theorem 2 now implies that $p_i^g(x) = p_i^f(F(x))$, i.e., the prices simply shift according to F.

There is one other important instance where we receive a simplified version of the formula, namely when the offset is not constant, but it only depends on its own level set, not on the individual point on the level set. This is important for the analysis of CFMMs with virtual reserves.

Theorem 3. *Let $g = f \circ F$ where $F(x) = x + v(x)$ and $v : \mathbb{R}^n \to \mathbb{R}^n$ is constant across each level set of g, i.e., $\forall x, x' : g(x) = g(x') \Rightarrow v(x) = v(x')$. Then*

$$p_i^g(x) = p_i^f(F(x))$$

Proof. To simplify the notation, assume WLOG that $n = 2$ and $i = 1$. Fix some reserve state x and let $k = g(x)$. The implicit function theorem tells us that there is some neighborhood U around x_1 and some differentiable function $\varphi : U \to \mathbb{R}$ such that $g(x_1', \varphi(x_1')) = g(x)\ \forall x_1' \in U$. We have $p_1^g(x) = -\frac{\mathrm{d}\varphi(x_1')}{\mathrm{d}x_1'}(x_1)$.

Consider the function $g'(x) := f(x + v(k))$ where $v(k)$ is treated as a constant independent of the input of g'. Let φ' be the implicit function implied by g'. Clearly, $g \neq g'$, but by choice of φ' we have $g(x_1', \varphi(x_1')) = g(x) = g'(x_1', \varphi(x_1'))$. By uniqueness, $\varphi = \varphi'$ in a neighborhood around x_1 and therefore also $-\frac{\mathrm{d}\varphi(x_1')}{\mathrm{d}x_1'}(x_1) = -\frac{\mathrm{d}\varphi'(x_1')}{\mathrm{d}x_1'}(x_1) = p_i^f(x + v(k)) = p_i^f(F(x))$. $\square$

Remark 2. One can apply Theorem 3 to transform a CFMM f into a variant with virtual reserves to achieve some property of the transformed prices (usually specific price bounds). To do this, one considers $v(k)$ as a function of the level set k, applies Theorem 3 to receive the transformed price and one can then often back out what v would have to be to achieve the desired properties.

To show that the resulting function $k \mapsto v(k)$ gives rise to a well-defined CFMM, one then has to show that the equation

$$k = f(x + v(k))$$

has a unique solution k for any x (among the permissible values of k; usually $k \geq 0$ is assumed). This is specific to the base CFMM f and the function $v(k)$.

The resulting function g is then rather complicated because it includes the computation of the above solution k. Specifically, let $k(x)$ be that solution. Then $g(x) = f(x + v(k(x)))$.

2.1 Examples

Scaling the CPMM.

Let $f(x, y) = xy$ be the 2-dimensional CPMM and let $F(t) := At := \lambda t$, where $\lambda > 0$ is some value and we write $t = (x, y)$ to make the notation more standard. As this is a linear transformation, we receive by Remark 1

$$p_x^g(t) = \frac{p^f(At) \cdot Ae_x}{p^f(At) \cdot Ae_y} = \frac{(\frac{\lambda y}{\lambda x}, 1) \cdot (\lambda, 0)}{(\frac{\lambda y}{\lambda x}, 1) \cdot (0, \lambda)} = \frac{\lambda \cdot y/x}{\lambda} = y/x = p_x^f(t).$$

Note that the prices have not changed due to this transformation! This may be intuitive given that we have scaled both reserves by the same amount.

Consider now a variant of this where $F(t) := At := (\lambda x,\, y)$, i.e., only the x dimension is scaled. We have

$$p_x^g(t) = \frac{p^f(At) \cdot Ae_x}{p^f(At) \cdot Ae_y} = \frac{(\frac{y}{\lambda x}, 1) \cdot (\lambda, 0)}{(\frac{y}{\lambda x}, 1) \cdot (0, 1)} = \frac{\lambda \cdot y/(\lambda x)}{1} = y/x = p_x^f(t).$$

Again, the prices have not changed. This may be surprising and one may have expected that the price p_x is either multiplied or divided by λ. However, we can see that this must hold by directly looking at the function $f \circ A$: we have $g(t) = f(At) = \lambda xy$. First note that the mechanism cannot "differentiate" between whether we scaled x or y. Formally, if $Bt := (x, \lambda y)$, then $f \circ A = f \circ B$. Therefore, also the prices must be equal for these two functions, so that the "direction" of the scaling must necessarily get lost. Second, note that $f(At) = k$ iff $f(t) = k/\lambda$. This implies that the level sets L_k are preserved by this transformation, even if their labels change. Formally, two points t, t' are in the same level set with respect to f iff they are in the same level set with respect to g.

The above preservation result may be surprising when one considers an operation where the denomination of (say) asset x is (say) "cents" while asset y is denoted in "dollars". Obviously, if we replace x by $100x$, then the price y/x changes by a factor $1/100$. However, the above result does not contradict this. Instead, what it means is that, if asset x is to be denoted in cents rather than dollars, one *can* express this by scaling. However, one can alternatively also simply use the prices provided by the CPMM that lie at a different (then usually very "unbalanced") part of the reserve space.

Note that the aforementioned invariance under asymmetric scaling is inherent to the CPMM and should not be expected from any other AMM. For instance, for the trivial constant-sum AMM, $f(t) = x + y$, scaling x by $\lambda \neq 1$ yields

$$p_x^g(t) = \frac{(1,1)\cdot(\lambda,0)}{(1,1)\cdot(0,1)} = \lambda \neq 1 = p_x^f(t) = p_x^f(At).$$

Here, no equivalence like the above exists.

It would be interesting to understand if the CPMM is the unique AMM with this strong form of "scale invariance" and how this relates to the folk theorem that the CPMM "spreads its liquidity equally over the whole range of prices"

References

Guillermo Angeris and Tarun Chitra. Improved price oracles: Constant function market makers. *Proceedings of the 2nd ACM Conference on Advances in Financial Technologies*, Oct 2020. URL `http://dx.doi.org/10.1145/3419614.3423251`.